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Translated by N. H. C.

UDC 62-50

ON AN ENCOUNTER GAME PROBLEM UNDER COMPOSITE CONTROLS

PMM Vol. 38, №3, 1974, pp. 402-408

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(Received October 30, 1973)

We examine a nonlinear differential game of the encounter of a conflict-controlled phase point with a given set. We prove sufficient conditions for the successful termination of the game in the class of mixed strategies. These conditions are based on the extremal construction introduced in [1] and modified here to conform to the question being discussed.

1. Statement of the problem. Let the motion of a controlled system be described by the differential equation

$$x^{\cdot} = f(t, x, u, v) \tag{1.1}$$

Here x is the system's n-dimensional phase vector; u and v are r-dimensional control vectors of the first and second players, respectively, and constrained by the conditions

$u \in P$, $v \in Q$, where P and Q are closed bounded sets. The vector-valued function $f(t, x, u, v)$ is continuous in all arguments, satisfies the Lipschitz condition in x in every bounded region, and satisfies the following continuability condition of the solutions of (1.1):

$$\| f(t, x, u, v) \| \leqslant \lambda (1 + \| x \|)$$

(λ is a constant, $\| p \|$ is the Euclidean norm of vector p). A closed set M is given in the space $\{x\}$. The initial position $\{t_0, x_0\}$ is fixed.

The first player's mixed strategies U, the second player's trivial strategy V_τ, and the motion $x[t] = x[t, t_0, x_0, U, V_\tau]$ of system (1.1) generated by them from the position $\{t_0, x_0\}$ are defined in accordance with [2]. The encounter game problem to be examined is formulated in the following manner.

Problem 1.1. Find the strategy U° for the position $\{t_0, x_0\}$ which at some instant $\vartheta > t_0$ ensures the fulfillment of the condition $x[\vartheta] \in M$ for every motion $x[t] = x[t, t_0, x_0, U^\circ, V_\tau]$.

2. The program problem. Let $\{\eta(du, dv)\}$ be the set of all regular Borel measures $\eta(du, dv)$ normed on $P \times Q$ and let $\{\eta_t(du, dv), [t_*, \vartheta]\}$ be the set of all functions $\eta_t(du, dv)$ defined on some time interval $[t_*, \vartheta]$, with values in $\{\eta(du, dv)\}$ and satisfying the condition that $\int\limits_P \int\limits_Q g(u, v)\, \eta_t(du, dv)$, is a measurable function on $[t_*, \vartheta]$ for any $g(u, v) \in C(P \times Q)$, where $C(P \times Q)$ is the space of real continuous functions defined on $P \times Q$ and having the norm $\| g \| = \sup\{| g(u, v) |, (u, v) \in P \times Q\}$.

By the symbol $L_1([t_*, \vartheta], C(P \times Q))$ we denote a Lebesgue space [3] of functions $h(t, u, v)$ defined and integrable on $[t_*, \vartheta]$, with values in $C(P \times Q)$ and having the norm

$$\| h \| = \int\limits_{t_*}^{\vartheta} \sup_{(u, v) \in P \times Q} | h(t, u, v) |\, dt$$

and by the symbol $L_1^*([t_*, \vartheta], C(P \times Q))$, the space adjoint to $L_1([t_*, \vartheta], C(P \times Q))$. We agree to treat the functions $\eta_t(du, dv)$ [3] as elements $\langle \eta, h \rangle$ of the space $L_1^*([t_*, \vartheta], C(P \times Q))$ of the form

$$\langle \eta, h \rangle = \iiint\limits_{t_* P Q}^{\vartheta} h(t, u, v)\, \eta_t(du, dv)\, dt$$

According to [3] the set $\{\eta_t(du, dv), [t_*, \vartheta]\}$ is compact in the $*$-weak topology of the space $L_1^*([t_*, \vartheta], C(P \times Q))$. Moreover, $\{\eta_t(du, dv), [t_*, \vartheta]\}$ is convex. The functions $\eta_t(du, dv)$ are called program controls. The program motions $x(t) = x(t, t_*, x_*, \eta_t)$ $(t_* \leqslant t \leqslant \vartheta)$ generated by the program controls $\eta_t(du, dv)$ from position $\{t_*, x_*\}$ are defined as the solutions of the equation

$$\dot{x}(t) = \iint\limits_{PQ} f(t, x(t), u, v)\, \eta_t(du, dv), \quad x(t_*) = x_* \tag{2.1}$$

Further, let $\{\mu(du)\}$ and $\{\nu(dv)\}$ be sets of all regular Borel measures $\mu(du)$ and $\nu(dv)$, normed on P and Q, respectively, and let $\{\mu_t(du), [t_*, \vartheta]\}$ and $\{\nu_t(dv), [t_*, \vartheta]\}$ be sets of functions $\mu_t(du)$ and $\nu_t(dv)$ defined on $[t_*, \vartheta]$, with values in $\{\mu(du)\}$ and $\{\nu(dv)\}$ and such that the functions

$$\int_P g'(u)\,\mu_t(du), \qquad \int_Q g''(v)\,\nu_t(dv)$$

are measurable on $[t_*, \vartheta]$ for any $g'(u) \in C(P)$ and $g''(v) \in C(Q)$.

We form all possible functions $\mu_t(du)\,\nu_t(dv)$ with $\mu_t(du) \in \{\mu_t(du), [t_*, \vartheta]\}$ and $\nu_t(dv) \in \{\nu_t(dv), [t_*, \vartheta]\}$. The functions $\mu_t(du)\,\nu_t(dv)$ also are program controls since they are defined on $[t_*, \vartheta]$ with values in $\{\eta(du, dv)\}$ and, according to [4], the function $\int_P\int_Q g(u, v)\,\mu_t(du)\,\nu_t(dv)$ is measurable on $[t_*, \vartheta]$ for $g(u, v) \in C(P \times Q))$.

A program $\{\eta_t(du, dv), [t_*, \vartheta]\}_\pi$ is any $*$-weakly closed set of program controls $\eta_t(du, dv)$, satisfying the conditions: (1) the program $\{\eta_t(du, dv), [t_*, \vartheta]\}_\pi$ contains at least one program control $\mu_t^*(du)\,\nu_t^*(dv)$; (2) if the program $\{\eta_t(du, dv), [t_*, \vartheta]\}_\pi$ contains the control $\mu_t^*(du)\,\nu_t^*(dv)$, it contains also all the controls $\mu_t(du)\,\nu_t^*(dv)$ for $\mu_t(du) \in \{\mu_t(du), [t_*, \vartheta]\}$. By virtue of the $*$-weak compactness of the set $\{\eta_t(du, dv), [t_*, \vartheta]\}$ the program $\{\eta_t(du, dv), [t_*, \vartheta]\}_\pi$ also is a $*$-weakly compact set.

We formulate the following program problem.

Problem 2.1. Given the position $\{t_*, x_*\}$ and the number $\vartheta > t_*$. Find the optimal program control $\eta_t^0(du, dv)$ satisfying the condition

$$\rho(x^\circ(\vartheta, t_*, x_*, \eta_t^\circ), M) = \max_{\{\eta_t\}_\pi} \min_{\eta_t \in \{\eta_t\}_\pi} \rho(x(\vartheta, t_*, x_*, \eta_t), M) \qquad (2.2)$$

($\rho(x, M)$ is the Euclidean distance from the point x to set M)

The existence of a solution of Problem 2.1 follows from the properties of program motions [3], the $*$-weak compactness of the programs, and the results in [1]. The motion $x^\circ(t) = x^\circ(t, t_*, x_*, \eta_t^\circ)$ $(t_* \leqslant t \leqslant \vartheta)$ generated by the optimal program control $\eta_t^\circ(du, dv)$ is called the optimal program motion. The program $\{\eta_t(du, dv), [t_*, \vartheta]\}_\pi$ giving the maximum in (2.2) and, obviously, containing the control $\eta_t^\circ(du, dv)$, is denoted by $\{\eta_t(du, dv), [t_*, \vartheta], x_*\}_\pi^\circ$ and is called the optimal program, while the $*$-weakly closed union of all optimal programs $\{\eta_t(du, dv), [t_*, \vartheta], x_*\}_\pi^\circ$ is denoted by $\{\eta_t(du, dv), [t_*, \vartheta], x_*\}_\pi^{\circ\circ}$ and is called the maximal optimal program.

3. Regularity conditions. We select a position $\{t_*, x_*\}$, $t_0 \leqslant t_* < \vartheta$, and a number τ, $t_* < \tau < \vartheta$. We form the set $\{\mu\nu^*\}$ of measures $\mu\nu^*$, where ν^* is some measure from $\{\nu\}$, and μ ranges over the whole set $\{\mu\}$. We consider the auxiliary motions $x^*(t) = x^*(t, t_*, x_*, \mu\nu^*)$ $(t_* \leqslant t \leqslant \tau)$, described by the equation

$$x^{*\cdot}(t) = \iint_{PQ} f(t_*, x_*, u, v)\,\mu(du)\,\nu^*(dv), \quad x^*(t_*) = x_* \qquad (3.1)$$

The final values $x^*(\tau)$ of motions (3.1) make up a set $X(\tau, \{t_*, x_*\}, \{\mu\nu^*\})$ which, by the properties of set $\{\mu\nu^*\}$, is closed, bounded, and convex.

We denote the right-hand side of (2.2) by the symbol $\varepsilon_0(t_*, x_*, \vartheta)$. Let the position $\{t_*, x_*\}$ be such that $\varepsilon_0(t_*, x_*, \vartheta) > 0$. We select some point $x^* \in X$. For this point we construct [1] a set $Y(x^*)$ of all points $x \in X$ for each of which we can find at least one program control η_t from $\{\eta_t, [\tau, \vartheta], x^*\}_\pi^{\circ\circ}$, i.e. the maximal program for the position $\{\tau, x^*\}$, such that the corresponding program motion $x(t) =$

$x(t, \tau, x^*; \eta_t)$ $(\tau \leqslant t \leqslant \vartheta)$ (2.1) satisfies the condition

$$\rho(x(\vartheta), M) \leqslant \varepsilon_0(t_*, x_*, \vartheta) + \varphi(\tau - t_*) \cdot (\tau - t_*)$$
$$\lim \varphi(\tau - t_*) = 0 \quad \text{as} \quad \tau \to t_*$$

We denote the closed convex hull of set $Y(x^*)$ by the symbol $Y^*(x^*)$. We formulate the following regularity condition for the game.

Condition 3.1. We say that the game is regular for some value of $\vartheta > t_0$ if for some sufficiently small number $\beta > 0$ for any position $\{t_*, x_*\}$, $t_0 \leqslant t_* < \vartheta$, for $\varepsilon_0(t_*, x_*, \vartheta) \in (0, \beta)$, for any set $\{\mu\nu^*\}$ and for any point $x^* \in X(\tau, \{t_*, x_*\}, \{\mu\nu^*\})$, $t_* < \tau < \vartheta$, we can find, for every point $x^{**} \in Y^*(x^*)$ a control $\eta_t \in \{\eta_t, [\tau, \vartheta], x^*\}_\pi^\infty$ such that the condition

$$\rho(x(\vartheta), M) \leqslant \varepsilon_0(t_*, x_*, \vartheta) + \varphi^*(\tau - t_*) \cdot (\tau - t_*)$$
$$\lim \varphi^*(\tau - t_*) = 0 \quad \text{as} \quad \tau \to t_*$$

is fulfilled uniformly in $x^* \in X(\tau, \{t_*, x_*\}, \{\mu\nu^*\})$ for the corresponding program motion $x(t) = x(t, \tau, x^{**}, \eta_t)$ $(\tau \leqslant t \leqslant \vartheta)$ (2.1).

By the symbol $Y_{min}(x^*)$ we denote the set of points $x^\circ \in X$ for which the relation

$$\min_{\eta_t \in \{\eta_t\}_\pi^{\infty}} \rho(x(\vartheta, \tau, x^\circ, \eta_t), M) = \min_{x \in X} \min_{\eta_t \in \{\eta_t\}_\pi^{\infty}} \rho(x(\vartheta, \tau, x, \eta_t), M)$$

is fulfilled, where the minimum is taken over all controls $\eta_t \in \{\eta_t, [\tau, \vartheta], x^*\}_\pi^\infty$. Let $Y_{\min}^*(x^*)$ be the closed convex hull of the set $Y_{\min}(x^*)$. We formulate another regularity condition for the game.

Condition 3.2. We say that the game is regular for certain value of $\vartheta > t_0$ if for some sufficiently small number $\beta > 0$, for any position $\{t_*, x_*\}$, $t_0 \leqslant t_* < \vartheta$, for $\varepsilon_0(t_*, x_*, \vartheta) \in (0, \beta)$, for any set $\{\mu\nu^*\}$, and for any point $x^* \in X(\tau, \{t_*, x_*\}, \{\mu\nu^*\})$, $t_* < \tau < \vartheta$ the condition

$$\gamma(Y^*_{\min}(x^*), Y_{\min}(x^*)) \leqslant \varphi_*(\tau - t_*) \cdot (\tau - t_*)$$
$$\lim \varphi_*(\tau - t_*) = 0 \quad \text{as} \quad \tau \to t_*$$

is fulfilled uniformly in $x^* \in X(\tau, \{t_*, x_*\}, \{\mu\nu^*\})$ and if the programs $\{\eta_t, [\tau, \vartheta], x^*\}_\pi^\infty$ are $*$-weakly continuous in x^* for $\varepsilon_0(\tau, x^*, \vartheta) \in (0, \beta)$. Here $\gamma(Y^*_{\min}(x^*), Y_{\min}(x^*))$ is the Hausdorff distance berween $Y^*_{\min}(x^*)$ and $Y_{\min}(x^*)$.

4. The set W_0 of program absorption. We consider the set $W(\varepsilon)$, $\varepsilon \geqslant 0$ of positions $\{t, x\}$ satisfying the condition $\varepsilon_0(t, x, \vartheta) \leqslant \varepsilon$, $t_0 \leqslant t \leqslant \vartheta$. The set $W(\varepsilon)$ is closed by virtue of the continuity of the function $\varepsilon_0(t, x, \vartheta)$ with respect to position $\{t, x\}$ (see [1]). Furthermore, under Condition3.1 or 3.2 and the condition $\varepsilon \in [0, \beta)$ the set $W(\varepsilon)$ possesses the property of strong $\tilde{u}$-stability [2, 5]. This property is stated as follows. A certain set W in a region $t_0 \leqslant t \leqslant \vartheta$ of space $\{t, x\}$ is said to be strongly $\tilde{u}$-stable if whatever be the position $\{t_*, x_*\} \in W$, $t_0 \leqslant t_* < \vartheta$, and the number $t^* \in (t_*, \vartheta]$, for any measure $\nu \in \{\nu\}$ we can find a motion $x[t] = x[t, t_*, x_*, \nu]$ $(t_* \leqslant t \leqslant t^*)$ for which the inclusion $\{t^*, x[t^*]\} \in W$ is fulfilled. Here $x[t] = x[t, t_*, x_*, \nu]$ is an absolutely continuous function satisfying the initial condition $x[t_*] = x_*$ and, for almost all $t_* \leqslant t \leqslant t^*$, the contingent equation [6]

$$x^{\cdot}[t] \in F(t, x[t]; \nu) \tag{4.1}$$

where $F(t, x; \nu)$ is the convex hull of all vectors f_u of the form

$$f_u = \int\limits_Q f(t, x, u, v)\,\nu(dv),\ u \in P.$$

The property of strong $\tilde{\alpha}$-stability of set $W(\varepsilon)$ for $\varepsilon \in (0, \beta)$ follows from lemmas stated below.

Lemma 4.1. Let the game be regular in the sense of Condition 3.1 for a selected value of $\vartheta > t_0$. Then whatever be the value of $\varepsilon \in (0, \beta)$, the position $\{t_*, x_*\}$ for which $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon$, $t_0 \leqslant t_* < \vartheta$, and the set $\{\mu\nu^*\}$, for any number $\alpha > 0$ we can find a number $\delta > 0$, $t_* + \delta \leqslant \vartheta$ such that for arbitrary $\tau_* \in (t_*, t_* + \delta]$ there exists at least one measure $\mu^*\nu^* \in \{\mu\nu^*\}$ which guarantees a motion $x^*(t) = x^*(t, t_*, x_*, \mu^*\nu^*)$ $(t_* \leqslant t \leqslant \tau_*)$ (3.1) satisfying the condition

$$\{\tau_*, x^*(\tau_*)\} \in W(\varepsilon + {}^1/_2\alpha\,(\tau_* - t_*)) \tag{4.2}$$

Proof. Assume that the lemma is incorrect. Then there exist a number $\varepsilon \in (0, \beta)$, a position $\{t_*, x_*\}$ for which $\varepsilon_0\ (t_*, x_*, \vartheta) = \varepsilon$, $t_0 \leqslant t_* < \vartheta$, and a set $\{\mu\nu^*\}$ such that we can find a number $\alpha > 0$ for which there exists, for any number $\delta > 0$, $t_* + \delta \leqslant \vartheta$, $\tau_* \in (t_*, t_* + \delta]$ such that all motions $x^*(t) = x^*(t, t_*, x_*, \mu\nu^*)$ $(t_* \leqslant t \leqslant \tau_*)$ (3.1) satisfy the condition

$$\{\tau_*, x^*(\tau_*)\} \notin W(\varepsilon + {}^1/_2\alpha\,(\tau_* - t_*)) \tag{4.3}$$

Consider the mapping $x^* \to Y^*(x^*)$. Under assumption (4.3) no element $x^* \in X$ whatsoever can be contained in its own image $Y^*(x^*)$ for the given $\tau_* \in (t_*, t_* + \delta]$. Let us choose $\delta > 0$ from the relation $\sup\{\varphi^*(t),\ t \in (0, \delta]\} \leqslant {}^1/_2\alpha$, where φ^* is the function occurring in Condition 3.1. We now assume to the contrary that some element x^* is contained in its own image $Y^*(x^*)$ for the indicated $\tau_* \in (t_*, t_* + \delta_*]$. Then by Condition 3.1 we can find a control $\eta_t \in \{\eta_t, [\tau_*, \vartheta], x^*\}_\pi^{\circ\circ}$ which guarantees the motion $x(t) = x(t, \tau_*, x^*, \eta_t)$ $(\tau_* \leqslant t \leqslant \vartheta)$ (2.1) satisfying the relation $\rho(x(\vartheta), M) \leqslant \varepsilon + \varphi^*(\tau_* - t_*)\cdot(\tau_* - t_*)$. By the choice of δ and τ_* the relation $\rho(x(\vartheta), M) \leqslant \varepsilon + {}^1/_2\ \alpha'(\tau_* - t_*)$ is valid for this same motion, which signifies the fulfillment of inclusion (4.2), but this contradicts (4.3). Hence it follows that no element $x^* \in X$ can be contained in its own image $Y^*(x^*)$. But the mapping $x^* \to Y^*(x^*)$ satisfies (see [1]) all the hypotheses of the theorem from [7] and, consequently, has at least one fixed point, i.e. there exists an element $x^* \in X$ satisfying the inclusion $x^* \in Y^*(x^*)$. However, as shown above, this inclusion is not possible. The contradiction obtained proves the lemma.

The next lemma relying on Condition 3.2 is proved analogously.

Lemma 4.2. Let the game be regular in the sense of Condition 3.2 for the chosen value of $\vartheta > t_0$. Then, whatever be the value of $\varepsilon \in (0, \beta)$, the position $\{t_*, x_*\}$ for which $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon$, $t_0 \leqslant t_* < \vartheta$, and the set $\{\mu\nu^*\}$, for any number $\alpha > 0$ we can find a number $\delta > 0$, $t_* + \delta \leqslant \vartheta$, such that for arbitrary $\tau_* \in (t_*, t_* + \delta]$ there exists at least one measure $\mu^*\nu^* \in \{\mu\nu^*\}$ which generates a motion $x^*(t) = x^*(t, t_*, x_*, \mu^*\nu^*)$ $(t_* \leqslant t \leqslant \tau_*)$ (3.1) satisfying condition (4.2).

Let us specify the motions $x^{(\alpha)}[t] = x^{(\alpha)}[t, t_*, x_*, \nu]$ $(t_* \leqslant t \leqslant t^*)$ as absolutely continuous functions satisfying the initial condition $x^{(\alpha)}[t_*] = x_*$ and, for almost all $t_* \leqslant t \leqslant t^*$, the contingent equation

$$x^{\cdot(\alpha)}[t] \in F^{(\alpha)}(t, x^{(\alpha)}[t];\ \nu) \tag{4.4}$$

where $F^{(\alpha)}(t, x; \nu)$ is the Euclidean α-neighborhood $(\alpha > 0)$ of the set $F(t, x; \nu)$ (4.1).

Lemma 4.3. Let the game be regular in the sense of Condition 3.1 or 3.2 for the chosen value of $\vartheta > t_0$. Then, whatever be the value of $\varepsilon \in (0, \beta)$, the position $\{t_*, x_*\}$ for which $\varepsilon_0(t_*, x_*, \vartheta) = \varepsilon$, $t_0 \leqslant t_* < \vartheta$, and the number $\alpha > 0$, for any measure $\nu^* \in \{\nu\}$ and any $t^* \in (t_*, \vartheta]$ we can find a motion $x^{(\alpha)}[t] = x^{(\alpha)}[t, t_*, x_*, \nu^*]$ $(t_* \leqslant t \leqslant t^*)$ (4.4), satisfying the condition

$$\{t^*, x^{(\alpha)}[t^*]\} \in W(\varepsilon + \alpha(t^* - t_*)) \tag{4.5}$$

Proof. Let $\tau_\alpha{}^\circ$ denote the greatest lower bound of the set of values of $\tau_\alpha \in (t_*, \vartheta]$ for each of which the condition

$$\{\tau_\alpha, x^{(\alpha)}[\tau_\alpha]\} \notin W(\varepsilon + \alpha(\tau_\alpha - t_*))$$

is fulfilled for all motions $x^{(\alpha)}[t] = x^{(\alpha)}[t, t_*, x_*, \nu^*]$ $(t_* \leqslant t \leqslant \tau_\alpha)$. Obviously, $\tau_\alpha{}^\circ > t_*$. Regardless of the lemma, let us assume that $\tau_\alpha{}^\circ < t^*$. As a consequence of the continuity of the function $\varepsilon_0(t, x, \vartheta)$ and of the compactness of the sheaf of motions $x^{(\alpha)}[t] = x^{(\alpha)}[t, t_*, x_*, \nu^*]$, for $t_* \leqslant t \leqslant \tau_\alpha{}^\circ$ we can find at least one motion $x_0{}^{(\alpha)}[t] = x_0{}^{(\alpha)}[t, t_*, x_*, \nu^*]$ from this sheaf, for which the inclusion

$$\{\tau_\alpha{}^\circ, x_0{}^{(\alpha)}[\tau_\alpha{}^\circ]\} \in W(\varepsilon + \alpha(\tau_\alpha{}^\circ - t_*))$$

is fulfilled. But, according to Lemma 4.1 (Lemma 4.2), for the position $\{\tau_\alpha{}^\circ, x_0{}^{(\alpha)}[\tau_\alpha{}^\circ]\}$, where $\varepsilon_0(\tau_\alpha{}^\circ, x_0{}^{(\alpha)}[\tau_\alpha{}^\circ], \vartheta) = \varepsilon_* \leqslant \varepsilon + \alpha(\tau_\alpha{}^\circ - t_*)$, we can find $\delta > 0$, $\tau_\alpha{}^\circ + \delta \leqslant \vartheta$ such that for every $\tau_* \in (\tau_\alpha{}^\circ, \tau_\alpha{}^\circ + \delta]$ there exists a motion $x^*(t) = x^*(t, \tau_\alpha{}^\circ, x_0{}^{(\alpha)}[\tau_\alpha{}^\circ], \mu^*\nu^*)$ $(\tau_\alpha{}^\circ \leqslant t \leqslant \tau_*)$ (3.1) satisfying the inclusion

$$\{\tau_*, x^*(\tau_*)\} \in W(\varepsilon_* + {}^1\!/_2\alpha(\tau_* - \tau_\alpha{}^\circ)) \tag{4.6}$$

Here $\delta > 0$ can be chosen so as to fulfil further the condition

$$\int\limits_P\int\limits_Q f(\tau_\alpha{}^\circ, x_0^{(\alpha)}[\tau_\alpha{}^\circ], u, v)\,\mu^*(du)\,\nu^*(dv) \in F^{(\alpha)}(\tau_*, x^{(\alpha)}[\tau_*]; \nu^*)$$

Then from inclusion (4.6) follows the inclusion

$$\{\tau_*, x^{(\alpha)}[\tau_*]\} \in W(\varepsilon + \alpha(\tau_* - t_*))$$

which contradicts the definition of $\tau_\alpha{}^\circ$. The contradiction obtained proves the lemma.

Lemma 4.4. Let the game be regular in the sense of Condition 3.1 or 3.2 for the selected value of $\vartheta > t_0$. Then the set $W(\varepsilon)$ is strongly $\tilde{u}$-stable for any value of $\varepsilon \in (0, \beta)$.

Proof. We choose a position $\{t_*, x_*\} \in W(\varepsilon)$, $t_0 \leqslant t_* < \vartheta$, $\varepsilon \in (0, \beta)$, a number $t^* \in (t_*, \vartheta]$, and a measure $\nu \in \{\nu\}$. We presecribe a sequence of numbers $\alpha_k > 0$ $(k = 1, 2, \ldots)$, $\lim \alpha_k = 0$ as $k \to \infty$. According to Lemma 4.3, for every number α_k we can find a motion $x^{(\alpha_k)}[t] = x^{(\alpha_k)}[t, t_*, x_*, \nu]$ $(t_* \leqslant t \leqslant t^*)$ (4.4) $(k = 1, 2, \ldots)$ satisfying the condition

$$\{t^*, x^{(\alpha_k)}[t^*]\} \in W(\varepsilon + \alpha_k(t^* - t_*)) \tag{4.7}$$

From the sequence of motions $x^{(\alpha_k)}[t] = x^{(\alpha_k)}[t, t_*, x_*, \nu]$ $(k = 1, 2, \ldots)$ we choose a subsequence converging uniformly to the motion $x^*[t] = x^*[t, t_*, x_*, \nu]$ $(t_* \leqslant t \leqslant t^*)$, being, obviously, the motion (4.1). But then, by virtue of (4.7), $\{t^*, x^*[t^*]\} \in W(\varepsilon)$

follows for the motion $x^*[t] = x^*[t, t_*, x_*, \nu]$. The lemma is proved.

By the symbol W_0 we denote set $W(\varepsilon)$ with $\varepsilon = 0$ and we call it the program absorption set. From Lemma 4.4 and [1] we obtain the following result for set W_0.

Lemma 4.5. Let the game be regular in the sense of Condition 3.1 or 3.2 for the chosen value of $\vartheta > t_0$. Then the program absorption set W_0 is strongly $\tilde{u}$-stable.

5. Basic theorems. The following statement is valid.

Theorem 5.1. Let the initial position $\{t_0, x_0\}$ belong to the program absorption set W_0 for some value of $\vartheta > t_0$ and let the game be regular in the sense of Condition 3.1 or 3.2. Then the strategy U° extremal to set W_0 solves Problem 1.1.

The validity of Theorem 5.1 follows directly from Lemma 4.5 and from [2, 8]; moreover, the strategy U° extremal to set W_0 is determined in the same way as in [2, 8].

Let us state one sufficient condition for the game's regularity in the sense of Condition 3.1.

Condition 5.1. We say that the game is strongly regular for some value of $\vartheta > t_0$ if the function $f(t, x, u, v)$ has continuous partial derivatives $\partial f / \partial x_i$ $(i = 1, \ldots, n)$ and if Problem 2.1 has the unique solution $\eta_t^\circ(du, dv)$ for any initial position $\{t_*, x_*\}$, $t_0 \leqslant t_* < \vartheta$, satisfying the condition $\varepsilon_0(t_*, x_*, \vartheta) \in (0, \beta)$, and here the point $m^\circ \in M$ nearest to the point $x^\circ(\vartheta) = x^\circ(\vartheta, t_*, x_*, \eta_t^\circ)$ is unique.

The following assertion is valid.

Lemma 5.1. If Condition 5.1 is fulfilled for some value of $\vartheta > t_0$, the game is regular in the sense of Condition 3.1 for this value of ϑ.

Lemma 5.1 can be proved by arguments analogous to those in [1].

The following theorem is a corollary of Lemma 5.1 and Theorem 5.1.

Theorem 5.2. Let the function $f(t, x, u, v)$ (1.1) have continuous partial derivatives $\partial f / \partial x_i$ $(i = 1, \ldots, n)$ and let the position $\{t_*, x_*\}$, $t_0 \leqslant t_* < \vartheta$ satisfy the condition $\varepsilon_0(t_*, x_*, \vartheta) \in (0, \beta)$ for some value of $\vartheta > t_0$, while the control $\eta_t^\circ(du, dv)$ solving Problem 2.1 for this position is unique and the point $m^\circ \in M$ nearest to the point $x^\circ(\vartheta) = x^\circ(\vartheta, t_*, x_*, \eta_t^\circ)$, is unique. Then, if the initial position $\{t_0, x_0\}$ belongs to the program absorption set W_0, the strategy U° extremal to set W_0 solves Problem 1.1.

The author thanks N.N. Krasovskii for statement of the problem and for valuable advice.

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Translated by N. H. C.

UDC 62-50

STOCHASTIC OPTIMAL CONTROL PROBLEMS WITH NONCONVEX CONSTRAINTS

PMM Vol. 38, №3, 1974, pp. 409-416
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(Received October 22, 1973)

We consider problems of the optimal control of a linear system subject to random actions. We assume that the system's phase coordinates are connected by nonconvex constraints which are necessarily also stochastic. We discuss deterministic problems equivalent to the stochastic ones mentioned. The unified approach to the problems formulated is based on the method for solving linear control systems, developed in [1] and modified in [2 – 4] for systems with constraints.

1. Let there be a control system

$$dx / dt = A(t)\ x(t) + B(t)\ u(t) + \xi(t) \tag{1.1}$$

where $x(t)$ is the n-dimensional phase coordinate vector, u is the r-dimensional control vector, $A(t)$, $B(t)$ are known continuous matrices of appropriate dimension, $\xi(t)$ is an n-dimensional vector-valued random process with specified probabilistic characteristics. The deterministic controls $u(t)$ (elements $u(\cdot)$) are chosen from a fixed weakly-compact convex set U of functions $u(t)$ from the r-vector space $L_2[t_\alpha,\ t_\beta]$. The control of the deterministic component

$$QX[t,\ t_\alpha]x^{(\alpha)} + Q\int_{t_\alpha}^{t} H[t, \tau]u(\tau)\,d\tau = y(t) \tag{1.2}$$

of the state vector $x(t)$ of system (1.1) is effected by choosing $u(\cdot)\in U$.

Problem 1.1. Given an initial state $x(t_\alpha) = x^{(\alpha)}$, a point $x^{(\beta)}$, a number $\varepsilon > 0$, a continuous function $\nu(t) > 0$ and an n-vector-valued function $x^\circ(t)$. From among the controls $u(\cdot)\in U$ find the $u^\circ(t)$ satisfying the condition $t_\beta^\circ - t_\alpha = \min$ under the constraints

$$M\rho_1[P(x(t_\beta) - x^{(\beta)})] \leqslant \varepsilon \tag{1.3}$$

$$M\rho_2[Q(x(t) - x^\circ(t))] \geqslant \nu(t), \quad t_\alpha \leqslant t \leqslant t_\beta \tag{1.4}$$

Here P, Q are known matrices of dimension $p \times n$, $q \times n$, respectively, $\rho_1[z_1]$, $\rho_2[z_2]$ are nonnegative convex functions in the spaces $R^{(p)}$, $R^{(q)}$ $(p,\ q \leqslant n)$, for